Polar Coding for Multi-Terminal Information Theory

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Goal

- Discuss polarization as a method for solving coding problems in information theory
- Pose some open problem areas

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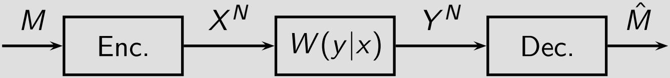
Outline

- Basic polar coding
- Polar coding for multi-terminal scenarios

Channel coding



- Reliability measure $P_e = P(\hat{M} \neq M)$
- Encoder maps message M to a codeword X^N of length N
- There are 2^{NR} messages, $H(M) = NR$
- Channel W is memoryless with capacity $C(W)$
- Shannon showed that P_e can be made arbitrarily small by selecting N large enough if (and only if) $R < C(W)$
- Shannon's method was non-constructive

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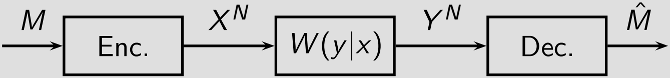

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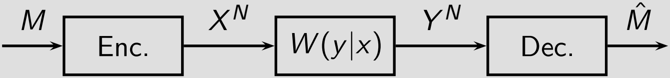

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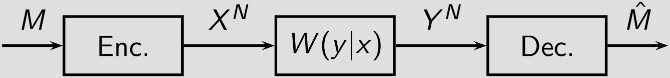

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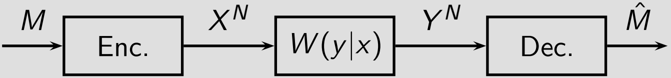

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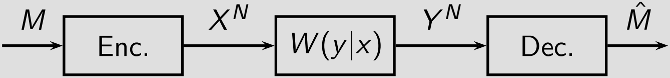

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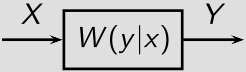
Channel capacity



- $(X, Y) \sim Q(x)W(y|x)$ an input-output pair for W
- Capacity

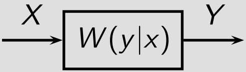
$$C(W) = \max_{Q} I(X;Y),$$

- Symmetric capacity

$$I(W) = I(X;Y),$$

with Q uniform

Channel capacity



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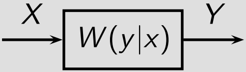

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Polar coding

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- an explicit construction
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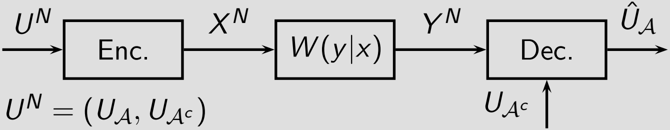
Basic polar coding method



- W is a channel with input alphabet $\{0, 1\}$
- Block length is $N = 2^n$ for some $n \geq 1$
- U^N and X^N are vectors of length N over $\{0, 1\}$
- U^N consists of a message part $U_{\mathcal{A}}$ and a frozen part $U_{\mathcal{A}^c}$
- Encoder computes $X^N = U^N G_N$ where

$$G_N = F^{\otimes n}, \qquad \text{with } F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- Decoder estimates $U_{\mathcal{A}}$ with knowledge of $U_{\mathcal{A}^c}$

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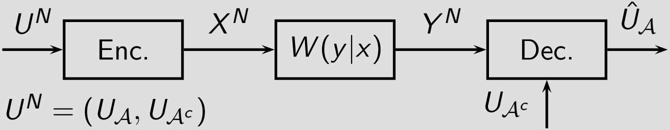

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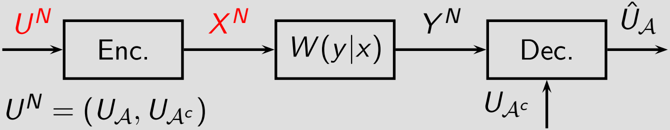

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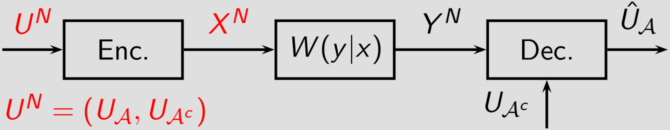

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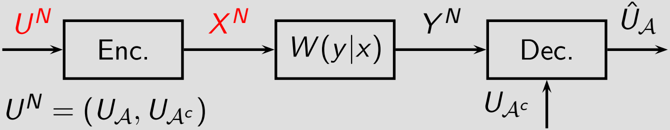

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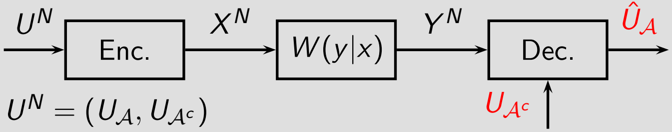

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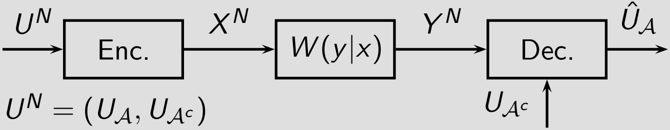

- Polarization refers to the possibility of selecting a set $\mathcal{A}$ of cardinality $|\mathcal{A}| = NI(W) - o(N)$ with $o(N)/N \to 0$ so that

$$H(U_i|Y^N, U^{i-1}) = O(2^{-N^{0.49}}), \quad \text{for all } i \in \mathcal{A},$$

  where $U^{i-1} = (U_1, \ldots, U_{i-1})$.

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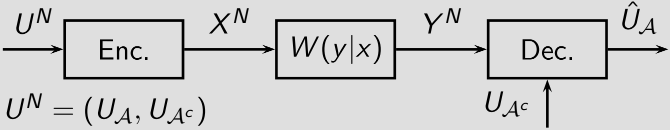

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Limitations of basic polar coding

Before polar codes can be considered as a candidate for solving all types of coding problems, a number of issues need to be addressed.

- Universality:
  A polar code good for one channel may not be good for another channel of the same or even higher capacity
- Capacity vs. symmetric capacity:
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  How to generalize the polar code construction to channels with non-binary input alphabets

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Universality issue

- If W is degraded wrt W', then a polar code good for W is also good for W'.
- But in general, a polar code good for W may be very bad for W' even if $I(W') > I(W)$.
- Şaşoğlu (2010) shows that if a polar code is designed for a Binary Symmetric Channel (BSC) W, then it is at least as good for any other channel W' with $I(W') \geq I(W)$ under ML decoding.
- So, the non-universal nature of polar coding is an artifact of the successive cancellation decoding method.
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Symmetric capacity issue

- Polar codes in their basic form achieve symmetric capacity $I(W)$ not the true capacity $C(W)$.
- All linear codes suffer from this defect and some general remedies exist.
- Korada (2009) proposed constructing polar codes over an extended alphabet and mapping extended alphabet to the binary alphabet to effect the desired distribution.
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Non-binary polar coding issue

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- Mori and Tanaka (2010) showed that for $q = p^m$, $p > 2$, the kernel

$$F = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}, \quad \gamma \neq 0, 1,$$

  polarizes channels with q-ary inputs.
- For channels for which the input alphabet size is not a prime power, there exist multi-level polarization approaches as discussed in Şaşoğlu et al. (2009), Park and Barg (2011,2012, 2013), and Sahebi and Pradhan (2011).

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Polarization theory and methods

Topic	References
Basic polarization	[28], [4], [1], [2]
General binary kernels	[29], [29], [30], [31], [32]
Non-binary kernels	[33], [30], [34], [35], [36], [37], [38]
Multi-level polarization	[39], [40], [41], [42],[43]
Universal polar codes	[44], [45], [46], [47]
Rate of polarization	[48], [49], [29]
Finite-length performance	[50], [51], [52],[53] ,[54], [55]
Code construction	[56], [57], [58], [59], [51]
Input distribution opt.	[2], [33], [60]

Recap of first part

Given a target rate R and an integer $q \geq 2$, there exist q-ary polar codes with length N and rate $> R$ such that, on any channel W with $C(W) > R$,

- encoding and decoding complexities are bounded by $O(N \log N)$,
- the probability of error is bounded by $O(e^{-N^{\beta}})$ for some $\beta > 0$ that depends on the channel W.

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- Polar coding for multi-terminal scenarios

A new approach

- The references cited use standard low-complexity properties of polar coding
- In each case, difficulty lies mainly in resolving complications arising from the need for universality, non-uniform input distributions, non-binary alphabets, etc.
- To explore the limits of polarization approach, it makes sense to use stronger polarizers unconstrained by complexity considerations.
- The primary question of interest is to see if polarization without complexity limits is as strong as random-coding/typical-set approach for proving achievability results.

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Random polarizers

Definition

A random polarizer is a one-to-one transformation $G_N : \mathcal{X}^N \to \mathcal{X}^N$ chosen at random (in equiprobable manner) from the class of all one-to-one functions from $\mathcal{X}^n$ to $\mathcal{X}^n$.

Polarization profile of a single random variable X

- Let X^N consist of N iid copies of a random variable over a finite alphabet $\mathcal{X}$.
- Let G_N be a random polarizer
- Let $U^N = G_N(X^N)$ be the transform of X^N
- The polarization profile of X^N is defined as the plot of $H(U_i|U^{i-1})$ vs. the normalized index $\hat{i} = i \log|\mathcal{X}|/N$.
- Polarization profile looks as follows:

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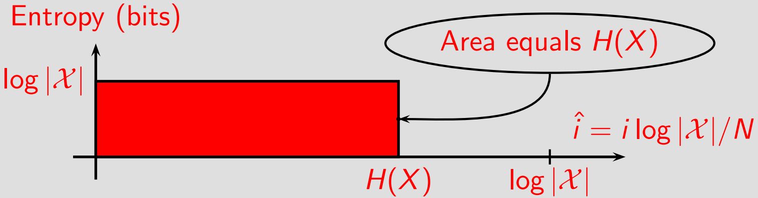

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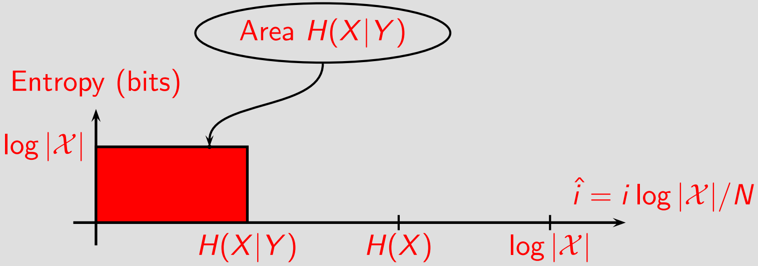

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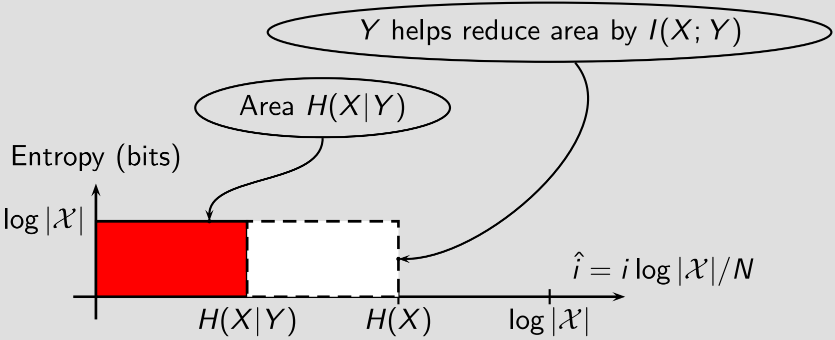
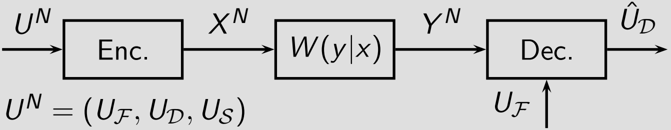


Achievability of channel capacity by random polarization



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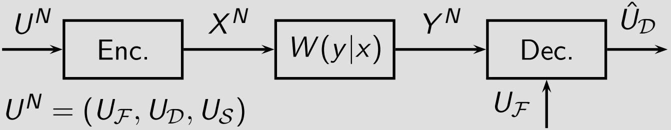

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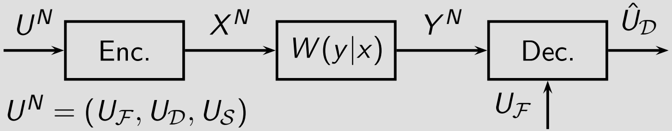

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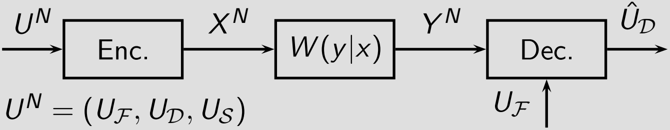

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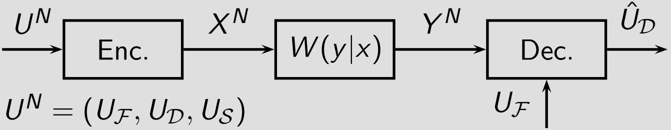

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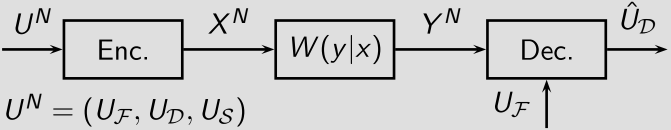

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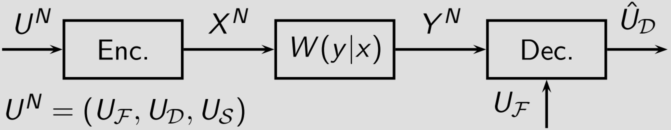

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Random polarization for two random variables (X_1, X_2)

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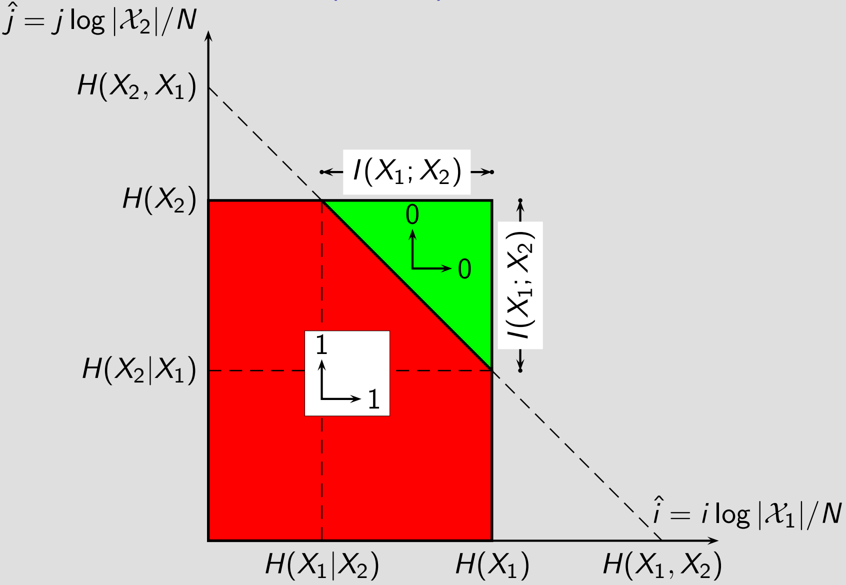

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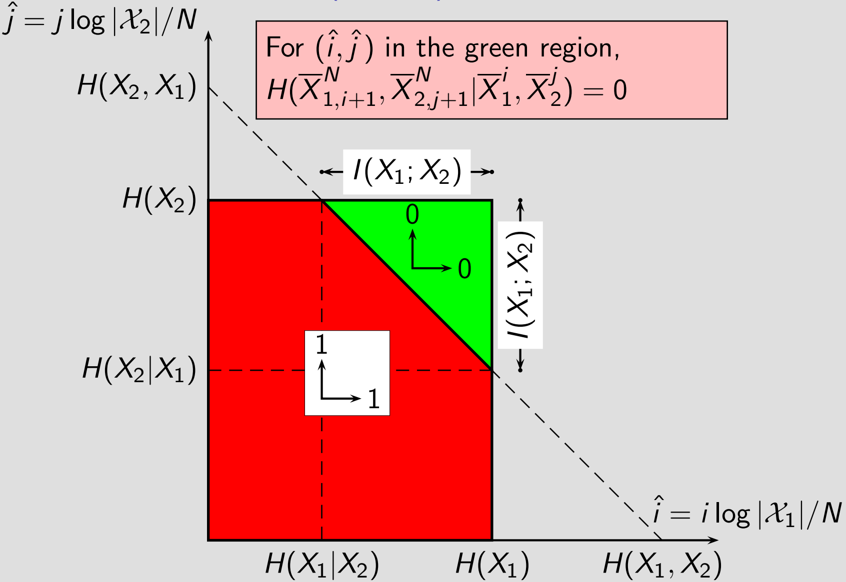

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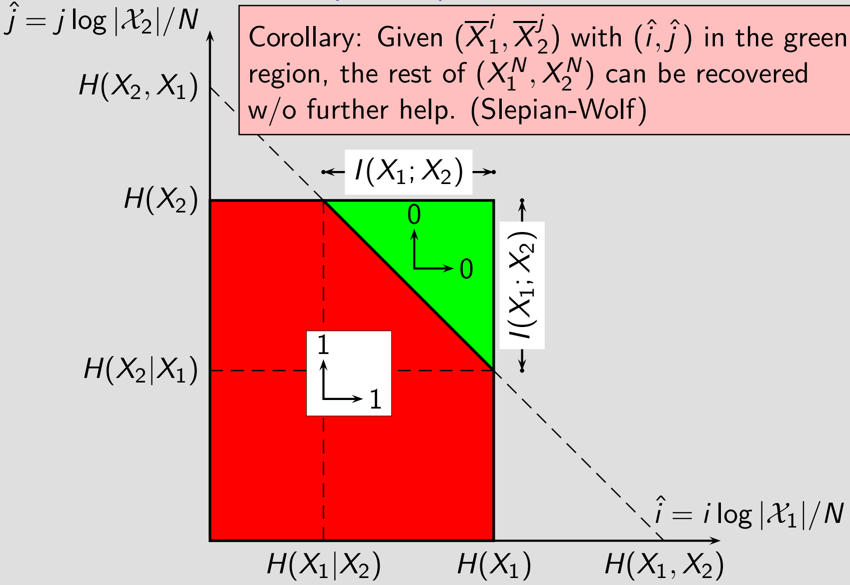
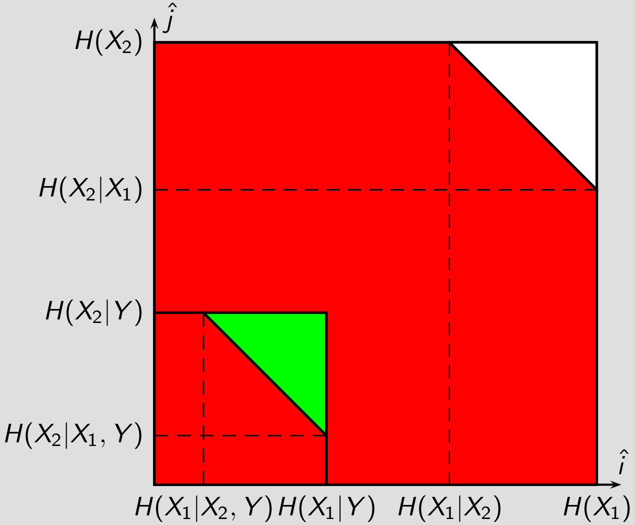
Corollary: Given $(\overline{X}_1^i, \overline{X}_2^j)$ with $(\hat{i}, \hat{j})$ in the green region, the rest of (X_1^N, X_2^N) can be recovered w/o further help. (Slepian-Wolf)



Polarization profile of (X_1, X_2) given Y



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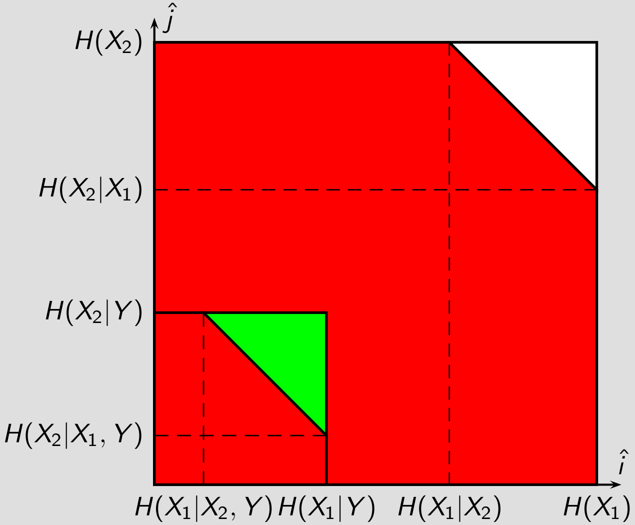
Starting from anywhere in the green region, a decoder can reach the top right corner.



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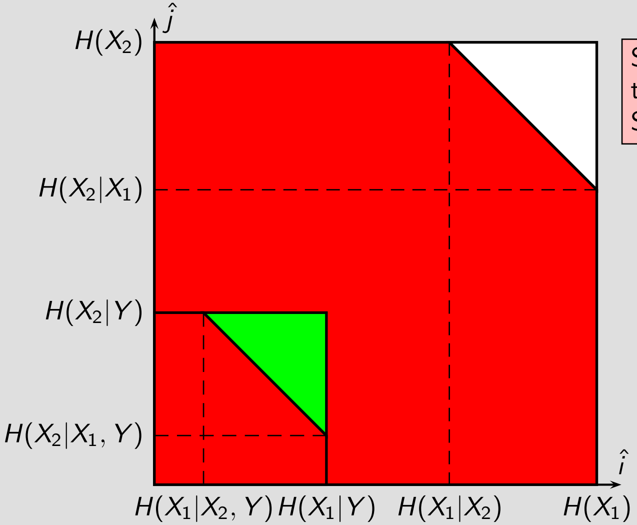

Step 1: Green is traversed by Slepian-Wolf.

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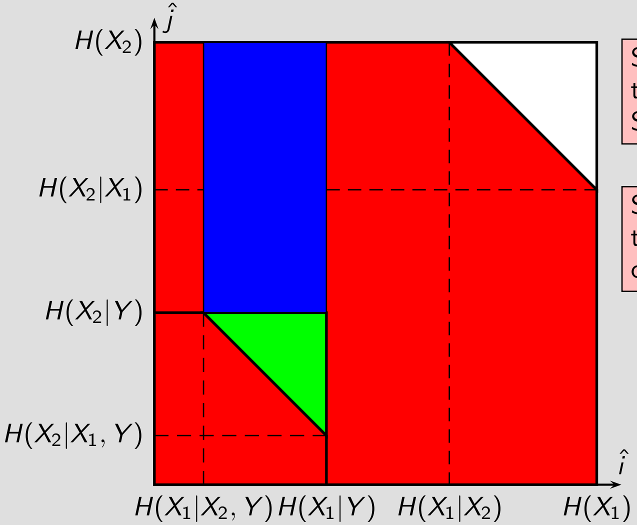

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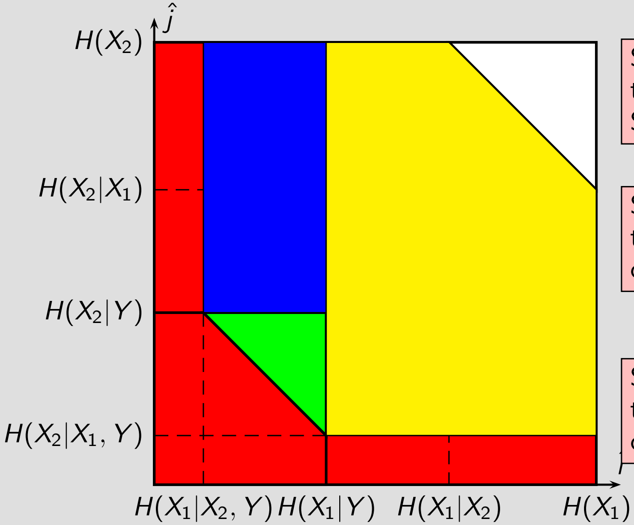
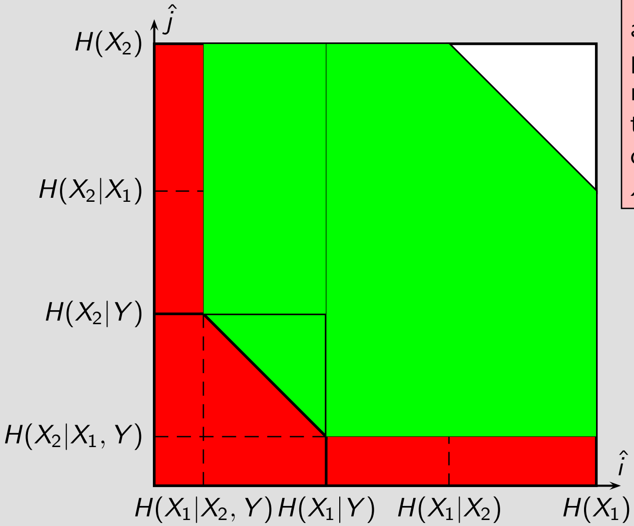


Step 1: Green is traversed by Slepian-Wolf.

Step 2: Blue is traversed by single-user channel coding.

Step 3: Yellow is traversed by single-user channel coding.

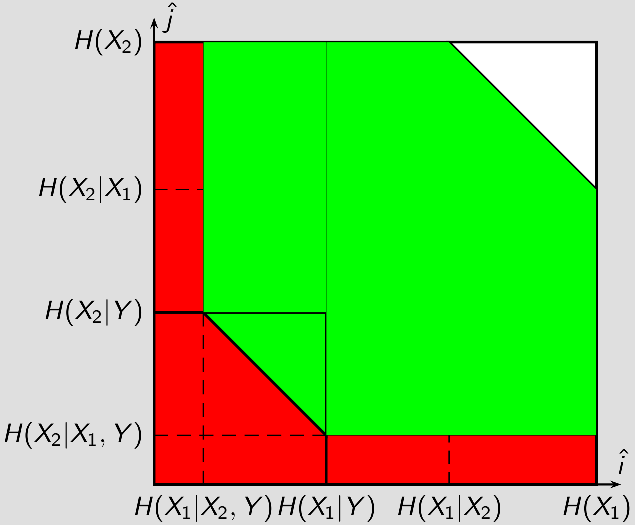
Polarization profile of (X_1, X_2) given Y



Top right corner is accessible from any point in the green region. The white triangle corresponds to common information in X_1 and X_2.

Polarization profile of (X_1, X_2) given Y

Achievable rates for a channel $(X_1, X_2) \to Y$ are obtained as
$R_1 < I(X_1; Y|X_2)$, $R_2 < I(X_2; Y|X_1)$, and $R_1 + R_2 < I(X_1, X_2; Y) - I(X_1; X_2)$



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Thank you!

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